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# THE CONE OF $\mathcal{Z}$ -TRANSFORMATIONS ON THE LORENTZ CONE\*

SÁNDOR Z. NÉMETH<sup>†</sup> AND M. SEETHARAMA GOWDA<sup>‡</sup>

**Abstract.** In this paper, we describe the structural properties of the cone of  $\mathcal{Z}$ -transformations on the Lorentz cone in terms of the semidefinite cone and copositive/completely positive cones induced by the Lorentz cone and its boundary. In particular, we describe its dual as a slice of the semidefinite cone as well as a slice of the completely positive cone of the Lorentz cone. This provides an example of an instance where a conic linear program on a completely positive cone is reduced to a problem on the semidefinite cone.

**Key words.**  $\mathcal{Z}$ -transformation, Lorentz cone, semidefinite cone, copositive cone, completely positive cone

**AMS subject classifications.** 90C33, 15A48

**1. Introduction.** Given a proper cone  $\mathcal{K}$  in a finite dimensional real Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ , a linear transformation  $A : \mathcal{H} \rightarrow \mathcal{H}$  is said to be a  $\mathcal{Z}$ -transformation on  $\mathcal{K}$  if

$$\left[ x \in \mathcal{K}, y \in \mathcal{K}^*, \text{ and } \langle x, y \rangle = 0 \right] \Rightarrow \langle Ax, y \rangle \leq 0,$$

where  $\mathcal{K}^*$  denotes the dual of  $\mathcal{K}$  in  $\mathcal{H}$ . Such transformations appear in various areas including economics, dynamical systems, optimization, see e.g., [3, 2, 12, 9] and the references therein. When  $\mathcal{H}$  is  $\mathbb{R}^n$  and  $\mathcal{K}$  is the nonnegative orthant,  $\mathcal{Z}$ -transformations become  $\mathcal{Z}$ -matrices, which are square matrices with nonpositive off-diagonal entries.

The set  $\mathcal{Z}(\mathcal{K})$  of all  $\mathcal{Z}$ -transformations on  $\mathcal{K}$  is a closed convex cone in the space of all (bounded) linear transformations on  $\mathcal{H}$ . Given their appearance and importance in various areas, describing/characterizing elements of  $\mathcal{Z}(\mathcal{K})$  and its interior, boundary, dual, etc., is of interest. An early result of Schneider and Vidyasagar [16] asserts that  $A$  is a  $\mathcal{Z}$ -transformation on  $\mathcal{K}$  if and only if  $e^{-tA}(\mathcal{K}) \subseteq \mathcal{K}$  for all  $t \geq 0$ ; consequently,

$$(1.1) \quad \mathcal{Z}(\mathcal{K}) = \overline{\mathbb{R}I - \pi(\mathcal{K})},$$

where  $\pi(\mathcal{K})$  denotes the set of all linear transformations that leave  $\mathcal{K}$  invariant,  $I$  denotes the identity transformation, and overline denotes the closure. To see another description of  $\mathcal{Z}(\mathcal{K})$ , let  $\text{LL}(\mathcal{K}) := \mathcal{Z}(\mathcal{K}) \cap -\mathcal{Z}(\mathcal{K})$  denote the lineality space of  $\mathcal{Z}(\mathcal{K})$ , the elements of which are called Lyapunov-like transformations. Then the inclusions

$$\mathbb{R}I - \pi(\mathcal{K}) \subseteq \text{LL}(\mathcal{K}) - \pi(\mathcal{K}) \subseteq \mathcal{Z}(\mathcal{K}) = \overline{\mathbb{R}I - \pi(\mathcal{K})}$$

imply that

$$\mathcal{Z}(\mathcal{K}) = \overline{\text{LL}(\mathcal{K}) - \pi(\mathcal{K})}.$$

As the cones  $\mathcal{Z}(\mathcal{K})$ ,  $\pi(\mathcal{K})$ , and  $\text{LL}(\mathcal{K})$  are generally difficult to describe for an arbitrary proper cone  $\mathcal{K}$ , we consider special cases. When  $\mathcal{K}$  is the nonnegative orthant,  $\mathcal{Z}(\mathcal{K})$  consists of square matrices with nonpositive off-diagonal entries,  $\pi(\mathcal{K})$  consists of nonnegative matrices, and  $\text{LL}(\mathcal{K})$  consists of diagonal matrices. Consequently, proper polyhedral cones can be handled via isomorphism arguments. Moving away from

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proper polyhedral cones, in this paper, we focus on the Lorentz cone (also called the ice-cream cone or the second-order cone as it is induced by the 2-norm) in the Euclidean space  $\mathbb{R}^n$ ,  $n > 1$ , defined by:

$$(1.2) \quad \mathcal{L} := \left\{ (t, u)^\top : t \in \mathbb{R}, u \in \mathbb{R}^{n-1}, t \geq \|u\| \right\}.$$

This, being an example of a symmetric cone, appears prominently in conic optimization [1]. For this cone, Stern and Wolkowicz [17] have shown that  $A \in \mathcal{Z}(\mathcal{L})$  if and only if for some real number  $\gamma$ , the matrix  $\gamma J - (JA + A^\top J)$  is positive semidefinite, where  $J$  is the diagonal matrix  $\text{diag}(1, -1, -1, \dots, -1)$ . Another result of Stern and Wolkowicz ([18], Theorem 4.2) asserts that

$$(1.3) \quad \mathcal{Z}(\mathcal{L}) = \text{LL}(\mathcal{L}) - \pi(\mathcal{L}).$$

(Going in the reverse direction, in a recent paper, Kuzma et al., [13] have shown that for an irreducible symmetric cone  $\mathcal{K}$ , the equality  $\mathcal{Z}(\mathcal{K}) = \text{LL}(\mathcal{K}) - \pi(\mathcal{K})$  holds only when  $\mathcal{K}$  is isomorphic to  $\mathcal{L}$ .) Characterizations of  $\pi(\mathcal{L})$  and  $\text{LL}(\mathcal{L})$  appear, respectively, in [14] and [20].

In this paper, we describe  $\mathcal{Z}(\mathcal{L})$  and its interior, boundary, and dual in terms of the semidefinite cone and the so-called copositive and completely positive cones induced by  $\mathcal{L}$  (or its boundary  $\partial(\mathcal{L})$ ), see below for the definitions. In particular, we describe the dual of  $\mathcal{Z}(\mathcal{L})$  as a slice of the semidefinite cone and also of the completely positive cone of  $\mathcal{L}$ . This provides an example of an instance where a conic linear optimization problem over a completely positive cone is reduced to a semidefinite problem. To elaborate, consider  $\mathbb{R}^n$ , the Euclidean  $n$ -space of (column) vectors with the usual inner product,  $\mathbb{R}^{n \times n}$ , the space of all real  $n \times n$  matrices with the inner product  $\langle X, Y \rangle = \text{tr}(X^\top Y)$ , and  $\mathcal{S}^n$ , the subspace of all real  $n \times n$  symmetric matrices in  $\mathbb{R}^{n \times n}$ . Corresponding to a closed cone  $\mathcal{C}$  (which is not necessarily convex) in  $\mathbb{R}^n$ , let

$$\mathcal{E}_{\mathcal{C}} := \text{copos}(\mathcal{C}) := \left\{ A \in \mathcal{S}^n : x^\top A x \geq 0 \text{ for all } x \text{ in } \mathcal{C} \right\}$$

denote the *copositive cone of  $\mathcal{C}$*  and

$$\mathcal{K}_{\mathcal{C}} := \text{compos}(\mathcal{C}) := \left\{ \sum_{u \in U} u u^\top : U \text{ is a finite subset of } \mathcal{C} \right\}$$

denote the *completely positive cone of  $\mathcal{C}$* . When  $\mathcal{C} = \mathbb{R}^n$ , these two cones coincide with the *semidefinite cone*  $\mathcal{S}_+^n$  (consisting of all real  $n \times n$  symmetric positive semidefinite matrices); when  $\mathcal{C} = \mathbb{R}_+^n$ , these reduce, respectively, to the (standard) copositive cone and completely positive cone. All these cones appear prominently in conic optimization. A result of Burer [5] (see also, [4, 7]) says that any nonconvex quadratic programming problem over a closed cone with additional linear and binary constraints can be reformulated as a linear program over a suitable completely positive cone. For this and other reasons, there is a strong interest in understanding copositive and completely positive cones. For the closed convex cones  $\mathcal{E}_{\mathcal{C}}$  and  $\mathcal{K}_{\mathcal{C}}$ , various structural properties (such as the interior, boundary) as well as duality, irreducibility, and homogeneity properties, have been investigated in the literature, see for example, [19, 6, 8, 11]. Taking  $\mathcal{C}$  to be one of  $\mathbb{R}^n$ ,  $\mathcal{L}$ , or  $\partial(\mathcal{L})$ , we show that

$$(1.4) \quad \mathcal{Z}(\mathcal{L})^* = \left\{ B \in \mathbb{R}^{n \times n} : \langle B, I \rangle = 0, -JB \in \mathcal{K}_{\mathcal{C}} \right\}$$

and deduce the equality of slices

$$(1.5) \quad \left\{ X \in \mathbb{R}^{n \times n} : \langle J, X \rangle = 0, X \in \mathcal{S}_+^n \right\} = \left\{ X \in \mathbb{R}^{n \times n} : \langle J, X \rangle = 0, X \in \mathcal{K}_{\mathcal{C}} \right\}.$$

**2. Preliminaries.** In a (finite dimensional real) Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ , a nonempty set  $\mathcal{C}$  is said to be a *closed cone* if it is closed and  $tx \in \mathcal{C}$  whenever  $x \in \mathcal{C}$  and  $t \geq 0$  in  $\mathbb{R}$ . Throughout this paper,  $\mathcal{C}$  denotes a closed cone. A nonempty set  $\mathcal{K}$  is said to be a *closed convex cone* if it is a closed cone which is also convex. Such a cone is said to be *proper* if  $\mathcal{K} \cap -\mathcal{K} = \{0\}$  and has nonempty interior. Corresponding to a closed convex cone  $\mathcal{K}$ , we define its dual in  $\mathcal{H}$  as the set

$$\mathcal{K}^* = \left\{ x \in \mathcal{H} : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K} \right\}.$$

We say that a linear transformation  $A : \mathcal{H} \rightarrow \mathcal{H}$  is *copositive* on  $\mathcal{K}$  if  $\langle Ax, x \rangle \geq 0$  for all  $x \in \mathcal{K}$ . We also let  $\pi(\mathcal{K}) = \{A : A(\mathcal{K}) \subseteq \mathcal{K}\}$ , where  $A$  denotes a linear transformation on  $\mathcal{H}$ . For a set  $S$  in  $\mathcal{H}$ , we denote the closure, interior, and the boundary by  $\overline{S}$ ,  $S^\circ$ , and  $\partial(S)$  respectively.

We will be considering closed convex cones in the space  $\mathcal{H} = \mathbb{R}^n$  which carries the usual inner product and in the space  $\mathbb{R}^{n \times n}$  which carries the inner product  $\langle X, Y \rangle := \text{tr}(X^\top Y)$ , where the trace of a square matrix is the sum of its diagonal entries. In  $\mathbb{R}^{n \times n}$ ,  $\mathcal{S}^n$  denotes the subspace of all symmetric matrices and  $\mathcal{A}^n$  denotes the subspace of all skew-symmetric matrices. We note that  $\mathbb{R}^{n \times n}$  is the orthogonal direct sum of  $\mathcal{S}^n$  and  $\mathcal{A}^n$ .

We recall some (easily verifiable) properties of the Lorentz cone  $\mathcal{L}$  given by (1.2).  $\mathcal{L}$  is a self-dual cone in  $\mathbb{R}^n$ , that is,  $\mathcal{L}^* = \mathcal{L}$ ; its interior and boundary are given, respectively, by

$$\mathcal{L}^\circ = \left\{ (t, u)^\top : t > \|u\| \right\},$$

$$\partial(\mathcal{L}) = \left\{ (t, u)^\top : t = \|u\| \right\} = \left\{ \alpha(1, u)^\top : \alpha \geq 0, \|u\| = 1 \right\}.$$

We also have

$$(2.1) \quad \left[ 0 \neq x, y \in \mathcal{L}, \langle x, y \rangle = 0 \right] \Rightarrow x = \alpha(1, u)^\top \text{ and } y = \beta(1, -u)^\top, \\ \text{for some } \alpha, \beta > 0 \text{ and } \|u\| = 1.$$

For a closed cone  $\mathcal{C}$  in  $\mathbb{R}^n$ , we consider the copositive cone  $\mathcal{E}_{\mathcal{C}}$  and the completely positive cone  $\mathcal{K}_{\mathcal{C}}$  (defined in the Introduction). Note that these are cones of symmetric matrices.

In the Hilbert space  $\mathcal{S}^n$  (which carries the inner product from  $\mathbb{R}^{n \times n}$ ), the following hold.

- (1)  $\mathcal{K}_{\mathcal{C}}$  is the dual cone of  $\mathcal{E}_{\mathcal{C}}$  [19].
- (2) When  $\mathcal{C} = \mathbb{R}^n$ , both  $\mathcal{E}_{\mathcal{C}}$  and  $\mathcal{K}_{\mathcal{C}}$  are proper cones ([10], Proposition 2.2). In particular, this holds when  $\mathcal{C}$  is one of  $\mathbb{R}^n$ ,  $\mathcal{L}$ , or  $\partial(\mathcal{L})$ .
- (3) We have  $\mathcal{S}_+^n = \mathcal{E}_{\mathbb{R}^n} \subset \mathcal{E}_{\mathcal{L}} \subset \mathcal{E}_{\partial(\mathcal{L})}$ , or equivalently,  $\mathcal{K}_{\partial(\mathcal{L})} \subset \mathcal{K}_{\mathcal{L}} \subset \mathcal{K}_{\mathbb{R}^n} = \mathcal{S}_+^n$ .

**3. Main results.** In this section, we provide a closure-free description of  $\mathcal{Z}(\mathcal{L})$  and, additionally, describe the dual, interior, and the boundary of  $\mathcal{Z}(\mathcal{L})$ . We recall that  $J = \text{diag}(1, -1, -1, \dots, -1)$  and  $\mathcal{A}^n$  denotes the set of all skew-symmetric matrices in  $\mathbb{R}^{n \times n}$ .

THEOREM 3.1. Let  $\mathcal{C}$  denote one of  $\mathbb{R}^n$ ,  $\mathcal{L}$ , or  $\partial(\mathcal{L})$ . Then,

$$(3.1) \quad \mathcal{Z}(\mathcal{L}) = \mathbb{R}I - J(\mathcal{E}_{\mathcal{L}} + \mathcal{A}^n).$$

*Proof.* Let  $A \in \mathcal{Z}(\mathcal{L})$ . From the result of Stern and Wolkowicz [17] mentioned in the Introduction, we have

$$2\gamma J - (JA + A^\top J) = 2P$$

for some  $\gamma \in \mathbb{R}$  and  $P \in \mathcal{S}_+^n$ . Hence,  $JA + (JA)^\top = 2(\gamma J - P)$ , which implies

$$(3.2) \quad 2JA = JA + (JA)^\top - [(JA)^\top - JA] = 2(\gamma J - P) - 2Q,$$

where  $2Q = (JA)^\top - JA$  is skew-symmetric. Since  $J^2 = I$ , this leads to

$$A = \gamma I - J(P + Q),$$

where  $P \in \mathcal{S}_+^n$  and  $Q \in \mathcal{A}^n$ . As  $\mathcal{S}_+^n \subset \mathcal{E}_{\mathcal{L}} \subset \mathcal{E}_{\partial(\mathcal{L})}$ , this proves that

$$(3.3) \quad \mathcal{Z}(\mathcal{L}) \subseteq \mathbb{R}I - J(\mathcal{S}_+^n + \mathcal{A}^n) \subseteq \mathbb{R}I - J(\mathcal{E}_{\mathcal{L}} + \mathcal{A}^n) \subseteq \mathbb{R}I - J(\mathcal{E}_{\partial(\mathcal{L})} + \mathcal{A}^n).$$

Now, to see the reverse inclusions, suppose  $A = \gamma I - J(P + Q)$  for some  $\gamma \in \mathbb{R}$ ,  $P \in \mathcal{E}_{\partial(\mathcal{L})}$ , and  $Q$  skew-symmetric. Let  $0 \neq x, y \in \mathcal{L}$  with  $\langle x, y \rangle = 0$ . By (2.1),  $x$  and  $y$  are in  $\partial(\mathcal{L})$ , and  $Jy$  is a positive multiple of  $x$ . Hence,  $\langle Px, Jy \rangle \geq 0$  as  $P \in \mathcal{E}_{\partial(\mathcal{L})}$  and  $\langle Qx, Jy \rangle = 0$  as  $Q$  is skew-symmetric. Thus,

$$\langle Ax, y \rangle = \gamma \langle x, y \rangle - \langle JPx, y \rangle + \langle JQx, y \rangle = -\langle Px, Jy \rangle + \langle Qx, Jy \rangle \leq 0.$$

This shows that  $A \in \mathcal{Z}(\mathcal{L})$  and so, inclusions in (3.3) turn into equalities. Thus we have (3.1).  $\square$

**Remarks.** From the above theorem, we have

$$\mathbb{R}I - J(\mathcal{S}_+^n + \mathcal{A}^n) = \mathbb{R}I - J(\mathcal{E}_{\mathcal{L}} + \mathcal{A}^n) = \mathbb{R}I - J(\mathcal{E}_{\partial(\mathcal{L})} + \mathcal{A}^n).$$

Multiplying throughout by  $J$  and noting  $-\mathcal{A}^n = \mathcal{A}^n$ , we get the equality of sets

$$(\mathbb{R}J - \mathcal{S}_+^n) + \mathcal{A}^n = (\mathbb{R}J - \mathcal{E}_{\mathcal{L}}) + \mathcal{A}^n = (\mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n,$$

where each set is a sum of  $\mathcal{A}^n$  and a subset of  $\mathcal{S}^n$ . Since  $\mathbb{R}^{n \times n} = \mathcal{S}^n + \mathcal{A}^n$  is an (orthogonal) direct sum decomposition, we see that

$$(3.4) \quad \mathbb{R}J - \mathcal{S}_+^n = \mathbb{R}J - \mathcal{E}_{\mathcal{L}} = \mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}.$$

These equalities can also be established via different arguments. A result of Loewy and Schneider [14] asserts that *A symmetric matrix  $X$  is copositive on  $\mathcal{L}$  if and only if there exists  $\mu \geq 0$  such that  $X - \mu J \in \mathcal{S}_+^n$* . (This is essentially a consequence of the so-called S-Lemma [15]: If  $A$  and  $B$  are two symmetric matrices with  $\langle Ax_0, x_0 \rangle > 0$  for some  $x_0$  and  $\langle Ax, x \rangle \geq 0 \Rightarrow \langle Bx, x \rangle \geq 0$ , then there exists  $\mu \geq 0$  such that  $B - \mu A$  is positive semidefinite.) This result gives the equality

$$\mathcal{E}_{\mathcal{L}} = \mathcal{S}_+^n + \mathbb{R}_+ J$$

and consequently  $\mathbb{R}J - \mathcal{S}_+^n = \mathbb{R}J - \mathcal{E}_{\mathcal{L}}$ . The equality

$$\mathcal{E}_{\partial(\mathcal{L})} = \mathcal{S}_+^n + \mathbb{R}J$$

can be seen via an application of Finsler' theorem [15] that says that if  $A$  and  $B$  are two symmetric matrices with  $[x \neq 0, \langle Ax, x \rangle = 0] \Rightarrow \langle Bx, x \rangle > 0$ , then there exists  $\mu \in \mathbb{R}$  such that  $B + \mu A$  is positive semidefinite. (For  $M \in \mathcal{E}_{\partial(\mathcal{L})}$  and vectors  $u, v \in \mathcal{L}^\circ$ , one has  $\langle Jx, x \rangle = 0 \Rightarrow \langle M_k x, x \rangle > 0$ , where  $k$  is a natural number and  $M_k := M + \frac{1}{k}uv^\top$ . When  $M_k + \mu_k J$  is positive semidefinite for all  $k$ , it follows that the sequence  $\mu_k$  is bounded. One can then use a limiting argument.) From this equality, one gets  $\mathbb{R}J - \mathcal{S}_+^n = \mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}$ .

Our next result deals with the dual of  $\mathcal{Z}(\mathcal{L})$ .

THEOREM 3.2. *Let  $\mathcal{C}$  denote one of  $\mathbb{R}^n$ ,  $\mathcal{L}$ , or  $\partial(\mathcal{L})$ . Then,*

$$\mathcal{Z}(\mathcal{L})^* = \left\{ B \in \mathbb{R}^{n \times n} : \langle B, I \rangle = 0, -JB \in \mathcal{K}_{\mathcal{C}} \right\}.$$

In particular, (1.5) holds.

*Proof.* We fix  $\mathcal{C}$ . From (3.1), we see that  $B \in \mathcal{Z}(\mathcal{L})^*$  if and only if

$$0 \leq \langle B, \gamma I - J(P + Q) \rangle$$

for all  $\gamma$  real,  $P$  in  $\mathcal{E}_{\mathcal{C}}$ , and  $Q$  in  $\mathcal{A}^n$ . Clearly, this holds if and only if

$$\langle B, I \rangle = 0, \langle -JB, P \rangle \geq 0, \text{ and } \langle -JB, Q \rangle = 0$$

for all  $\gamma$ ,  $P$ , and  $Q$  specified above. Now, with the observation that a (real) matrix is orthogonal to all skew-symmetric matrices in  $\mathbb{R}^{n \times n}$  if and only if it is symmetric, this further simplifies to

$$\langle B, I \rangle = 0 \text{ and } -JB \in \mathcal{E}_{\mathcal{C}}^*,$$

where  $\mathcal{E}_{\mathcal{C}}^*$  is the dual of  $\mathcal{E}_{\mathcal{C}}$  computed in  $\mathcal{S}^n$ . Since  $\mathcal{K}_{\mathcal{C}} = \mathcal{E}_{\mathcal{C}}^*$  in  $\mathcal{S}^n$ , we see that  $B \in \mathcal{Z}(\mathcal{L})^*$  if and only if  $\langle B, I \rangle = 0$  and  $-JB \in \mathcal{K}_{\mathcal{C}}$ . This completes the proof.  $\square$

We remark that (1.5) can be deduced directly from (3.4) by taking the duals in  $\mathcal{S}^n$ .

In our final result, we describe the interior and boundary of  $\mathcal{Z}(\mathcal{L})$ . First, we recall some definitions from [9]. Let

$$\Omega := \left\{ (x, y) \in \mathcal{L} \times \mathcal{L} : \|x\| = 1 = \|y\| \text{ and } \langle x, y \rangle = 0 \right\}.$$

It is easy to see that  $\Omega$  is compact and, from (2.1),

$$(3.5) \quad \Omega = \left\{ (x, Jx) : x \in \partial(\mathcal{L}), \|x\| = 1 \right\}.$$

For any  $A \in \mathbb{R}^{n \times n}$ , let

$$\gamma(A) := \max \left\{ \langle Ax, y \rangle : (x, y) \in \Omega \right\}.$$

Note that  $A \in \mathcal{Z}(\mathcal{L})$  if and only if  $\gamma(A) \leq 0$ . We say that  $A \in \mathbb{R}^{n \times n}$  is a *strict- $\mathcal{Z}$ -transformation on  $\mathcal{L}$*  if

$$[0 \neq x, y \in \mathcal{L}, \langle x, y \rangle = 0] \Rightarrow \langle Ax, y \rangle < 0.$$

The set of all such transformations is denoted by  $\text{str}(\mathcal{Z}(\mathcal{L}))$ . For  $A \in \mathbb{R}^{n \times n}$ , the following statements are shown in [9], Theorem 3.1:

$$\gamma(A) < 0 \iff A \in \mathcal{Z}(\mathcal{L})^\circ \iff A \in \text{str}(\mathcal{Z}(\mathcal{L}))$$

and

$$\gamma(A) = 0 \iff A \in \partial(\mathcal{Z}(\mathcal{L})).$$

Recall that  $\mathcal{E}_{\mathcal{L}}$  consists of all symmetric matrices that are copositive on  $\mathcal{L}$ . We say that a symmetric matrix  $P$  is *strictly copositive on  $\mathcal{L}$*  if  $0 \neq x \in \mathcal{L} \Rightarrow \langle Px, x \rangle > 0$ ; the set of all such matrices is denoted by  $\text{str}(\mathcal{E}_{\mathcal{L}})$ . Similarly, one defines  $\text{str}(\mathcal{E}_{\partial(\mathcal{L})})$ .

**COROLLARY 3.3.** *The following statements hold:*

$$\mathcal{Z}(\mathcal{L})^\circ = \text{str}(\mathcal{Z}(\mathcal{L})) = \mathbb{R}I - J\left(\text{str}(\mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n\right)$$

and

$$\partial(\mathcal{Z}(\mathcal{L})) = \mathbb{R}I - J\left(\partial_*(\mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n\right),$$

where  $\partial_*(\mathcal{E}_{\partial(\mathcal{L})})$  denotes the boundary of  $\mathcal{E}_{\partial(\mathcal{L})}$  in  $\mathcal{S}^n$ .

*Proof.* We first deal with the interior of  $\mathcal{Z}(\mathcal{L})$ . The equality

$$\{A \in \mathbb{R}^{n \times n} : \gamma(A) < 0\} = \mathcal{Z}(\mathcal{L})^\circ = \text{str}(\mathcal{Z}(\mathcal{L}))$$

has already been observed in [9], Theorem 3.1. To see the first assertion, we show that  $\gamma(A) < 0$  if and only if  $A = \theta I - J(P + Q)$  for some  $\theta \in \mathbb{R}$ ,  $P$  (symmetric) strictly copositive on  $\partial(\mathcal{L})$ , and  $Q$  skew-symmetric. Suppose  $\gamma(A) < 0$ . Then, for any  $\theta \in \mathbb{R}$ ,

$$\max \left\{ \langle (A - \theta I)x, y \rangle : (x, y) \in \Omega \right\} < 0,$$

which, from (3.5) becomes

$$\min \left\{ \langle J(\theta I - A)x, x \rangle : x \in \partial(\mathcal{L}), \|x\| = 1 \right\} > 0.$$

Now, fix  $\theta$  and let  $J(\theta I - A) = P + Q$ , where  $P \in \mathcal{S}^n$  and  $Q \in \mathcal{A}^n$ . As  $\langle Qx, x \rangle = 0$  for any  $x$ , the above inequality implies that  $\min \left\{ \langle Px, x \rangle : x \in \partial(\mathcal{L}), \|x\| = 1 \right\} > 0$ .

This proves that  $P$  is strictly copositive on  $\partial(\mathcal{L})$ . Rewriting  $J(\theta I - A) = P + Q$ , we see that  $A = \theta I - J(P + Q)$  which is of the required form.

To see the converse, suppose  $A = \theta I - J(P + Q)$ , where  $\theta \in \mathbb{R}$ ,  $P$  (symmetric) strictly copositive on  $\partial(\mathcal{L})$ , and  $Q$  skew-symmetric. Using (3.5), we can easily verify that  $\gamma(A) < 0$ . Thus,  $A \in \text{str}(\mathcal{Z}(\mathcal{L}))$ .

An argument similar to the above will show that  $\gamma(A) = 0$  if and only if  $A = \theta I - J(P + Q)$  for some  $\theta \in \mathbb{R}$ ,  $P \in \partial_*(\mathcal{E}_{\partial(\mathcal{L})})$ , and  $Q$  skew-symmetric. This gives the statement regarding the boundary of  $\mathcal{Z}(\mathcal{L})$ .  $\square$

We end the paper with a remark dealing with conic linear programs. Motivated by the result of Burer (mentioned in the Introduction), we consider a conic linear program on a completely positive cone  $\mathcal{K}_{\mathcal{C}}$  (where  $\mathcal{C}$  is a closed cone):

$$\min \left\{ \langle c, x \rangle : Ax = b, x \in \mathcal{K}_{\mathcal{C}} \right\}.$$

While such a problem is generally hard to solve, we ask: (When) can we replace  $\mathcal{K}_{\mathcal{C}}$  by  $\mathcal{S}_+^n$  and thus reduce the above problem to the semidefinite programming problem  $\min \left\{ \langle c, x \rangle : Ax = b, x \in \mathcal{S}_+^n \right\}$ ? Just replacing  $\mathcal{K}_{\mathcal{C}}$  by  $\mathcal{S}_+^n$  without handling the constraint  $Ax = b$  is not viable as  $\mathcal{K}_{\mathcal{C}} = \mathcal{S}_+^n$  if and only if  $\mathcal{C} \cup -\mathcal{C} = \mathbb{R}^n$  (which fails to hold when  $n > 1$  and  $\mathcal{C}$  is pointed), see [11]. While we do not answer this broad question, we point out, as a consequence of (1.5) that for any  $C \in \mathcal{S}^n$ ,

$$\min \left\{ \langle C, X \rangle : \langle X, J \rangle = 0, X \in \mathcal{K}_{\mathcal{C}} \right\} = \min \left\{ \langle C, X \rangle : \langle X, J \rangle = 0, X \in \mathcal{S}_+^n \right\}.$$

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